22117203

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 1

Wednesday 4 May 2011 (afternoon)

2 hours

Candidate session number

0	0							

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 7]*

Events A and B are such that $\mathrm{P}(A) = 0.3$ and $\mathrm{P}(B) = 0.4$.

(a) Find the value of $\mathrm{P}(A \cup B)$ when

(i) A and B are mutually exclusive;

(ii) A and B are independent. *[4 marks]*

(b) Given that $\mathrm{P}(A \cup B) = 0.6$, find $\mathrm{P}(A \mid B)$. *[3 marks]*

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2. *[Maximum mark: 4]*

Given that $\frac{z}{z+2} = 2 - \mathrm{i}$, $z \in \mathbb{C}$, find z in the form $a + \mathrm{i}b$.

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3. *[Maximum mark: 7]*

A geometric sequence u_1, u_2, u_3, ... has $u_1 = 27$ and a sum to infinity of $\frac{81}{2}$.

(a) Find the common ratio of the geometric sequence. *[2 marks]*

An arithmetic sequence v_1, v_2, v_3, ... is such that $v_2 = u_2$ and $v_4 = u_4$.

(b) Find the greatest value of N such that $\sum_{n=1}^{N} v_n > 0$. *[5 marks]*

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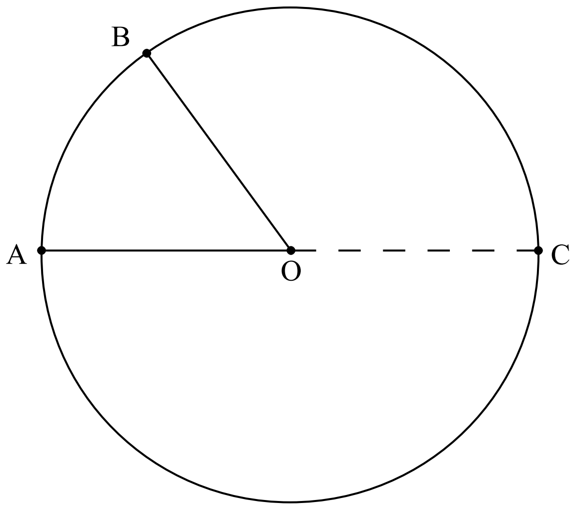
4. *[Maximum mark: 5]*

The diagram below shows a circle with centre O. The points A, B, C lie on the circumference of the circle and [AC] is a diameter.



Let $\vec{OA} = \boldsymbol{a}$ and $\vec{OB} = \boldsymbol{b}$.

(a) Write down expressions for $\vec{AB}$ and $\vec{CB}$ in terms of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. *[2 marks]*

(b) Hence prove that angle $A\hat{B}C$ is a right angle. *[3 marks]*

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5. *[Maximum mark: 5]*

(a) Show that $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$. *[2 marks]*

(b) Hence find the value of $\cot\frac{\pi}{8}$ in the form $a+b\sqrt{2}$, where $a, b \in \mathbb{Z}$. *[3 marks]*

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6. *[Maximum mark: 5]*

In a population of rabbits, 1 % are known to have a particular disease. A test is developed for the disease that gives a positive result for a rabbit that **does** have the disease in 99 % of cases. It is also known that the test gives a positive result for a rabbit that **does not** have the disease in 0.1 % of cases. A rabbit is chosen at random from the population.

(a) Find the probability that the rabbit tests positive for the disease. *[2 marks]*

(b) Given that the rabbit tests positive for the disease, show that the probability that the rabbit does not have the disease is less than 10 %. *[3 marks]*

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7. *[Maximum mark: 6]*

Find the area enclosed by the curve $y = \arctan x$, the x-axis and the line $x = \sqrt{3}$.

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8. *[Maximum mark: 6]*

Consider the functions given below.

$$f(x) = 2x + 3$$

$$g(x) = \frac{1}{x},\ x \neq 0$$

(a) (i) Find $(g \circ f)(x)$ and write down the domain of the function.

(ii) Find $(f \circ g)(x)$ and write down the domain of the function. *[2 marks]*

(b) Find the coordinates of the point where the graph of $y = f(x)$ and the graph of $y = (g^{-1} \circ f \circ g)(x)$ intersect. *[4 marks]*

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9. *[Maximum mark: 7]*

Show that the points $(0, 0)$ and $\left(\sqrt{2\pi}, -\sqrt{2\pi}\right)$ on the curve $e^{(x+y)} = \cos(xy)$ have a common tangent.

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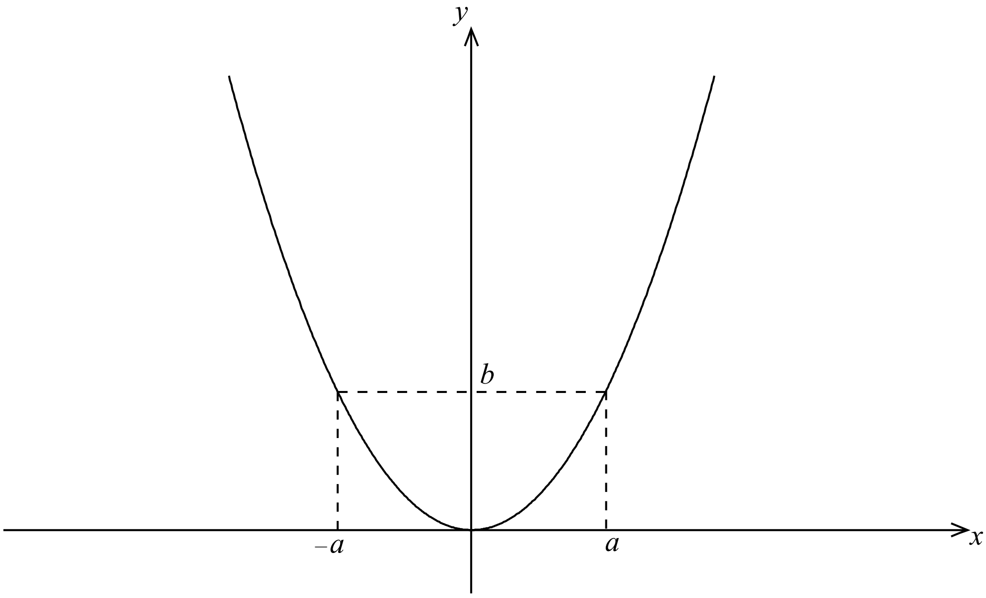
10. *[Maximum mark: 8]*

The diagram below shows the graph of the function $y = f(x)$, defined for all $x \in \mathbb{R}$, where $b > a > 0$.



Consider the function $g(x) = \dfrac{1}{f(x-a)-b}$.

(a) Find the largest possible domain of the function g. *[2 marks]*

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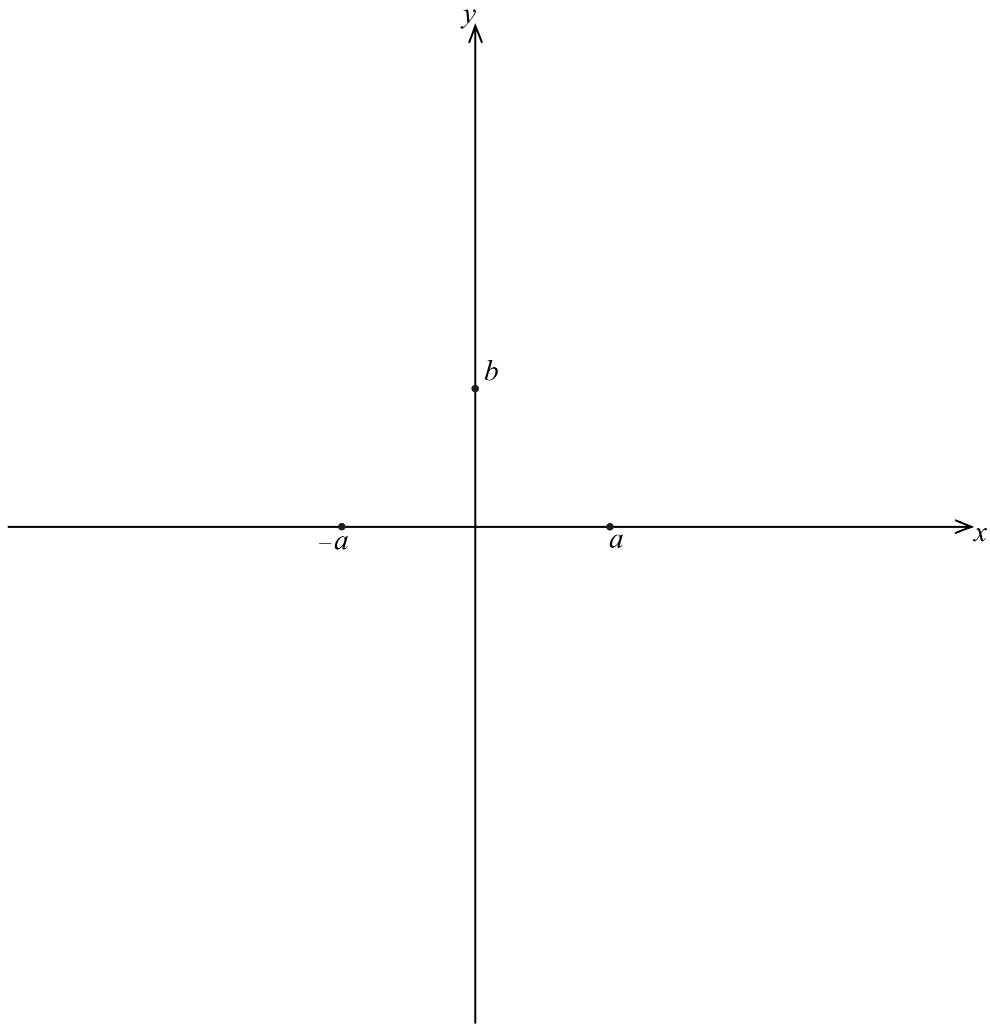
(Question 10 continued)

(b) On the axes below, sketch the graph of $y = g(x)$. On the graph, indicate any asymptotes and local maxima or minima, and write down their equations and coordinates. *[6 marks]*



*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

11. *[Maximum mark: 19]*

The points $A(1, 2, 1)$, $B(-3, 1, 4)$, $C(5, -1, 2)$ and $D(5, 3, 7)$ are the vertices of a tetrahedron.

(a) Find the vectors $\vec{AB}$ and $\vec{AC}$. *[2 marks]*

(b) Find the Cartesian equation of the plane Π that contains the face ABC. *[4 marks]*

(c) Find the vector equation of the line that passes through D and is perpendicular to Π. Hence, or otherwise, calculate the shortest distance to D from Π. *[5 marks]*

(d) (i) Calculate the area of the triangle ABC.

(ii) Calculate the volume of the tetrahedron ABCD. *[4 marks]*

(e) Determine which of the vertices B or D is closer to its opposite face. *[4 marks]*

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

12. *[Maximum mark: 19]*

Consider the function $f(x)=\dfrac{\ln x}{x}$, $0<x<\mathrm{e}^2$.

(a) (i) Solve the equation $f'(x)=0$.

(ii) Hence show the graph of f has a local maximum.

(iii) Write down the range of the function f. *[5 marks]*

(b) Show that there is a point of inflexion on the graph and determine its coordinates. *[5 marks]*

(c) Sketch the graph of $y=f(x)$, indicating clearly the asymptote, x-intercept and the local maximum. *[3 marks]*

(d) Now consider the functions $g(x)=\dfrac{\ln|x|}{x}$ and $h(x)=\dfrac{\ln|x|}{|x|}$, where $0<|x|<\mathrm{e}^2$.

(i) Sketch the graph of $y=g(x)$.

(ii) Write down the range of g.

(iii) Find the values of x such that $h(x)>g(x)$. *[6 marks]*

13. *[Maximum mark: 22]*

(a) Write down the expansion of $(\cos\theta+\mathrm{i}\sin\theta)^3$ in the form $a+\mathrm{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$. *[2 marks]*

(b) Hence show that $\cos 3\theta=4\cos^3\theta-3\cos\theta$. *[3 marks]*

(c) Similarly show that $\cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$. *[3 marks]*

(d) **Hence** solve the equation $\cos 5\theta+\cos 3\theta+\cos\theta=0$, where $\theta\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$. *[6 marks]*

(e) By considering the solutions of the equation $\cos 5\theta=0$, show that $\cos\dfrac{\pi}{10}=\sqrt{\dfrac{5+\sqrt{5}}{8}}$ and state the value of $\cos\dfrac{7\pi}{10}$. *[8 marks]*